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Torsional surface waves in a gradient-elastic half-space

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Abstract

The present work deals with torsional wave propagation in a linear gradient-elastic half-space. More specifically, we prove that torsional surface waves (i.e. waves with amplitudes exponentially decaying with distance from the free surface) do exist in a homogeneous gradient-elastic half-space. This finding is in contrast with the well-known result of the classical theory of linear elasticity that torsional surface waves do not exist in a homogeneous half-space. The weakness of the classical theory, at this point, is only circumvented by modeling the half-space as having material properties variable with depth (E. Meissner, Elastische Oberflachenwellen mit Dispersion in einem inhomogenen Medium, Vierteljahrsschrift der Naturforschenden Gesellschaft in Zurich 66 (1921) 181–195; I. Vardoulakis, Torsional surface waves in inhomogeneous elastic media, Internat. J. Numer. Anal. Methods Geomech. 8 (1984) 287–296; G.A. Maugin, Shear horizontal surface acoustic waves on solids, in: D.F. Parker, G.A. Maugin (Eds.), Recent Developments in Surface Acoustic Waves, Springer Series on Wave Phenomena, vol. 7, Springer, Berlin, 1988, pp. 158–172), as a layered structure (Maugin, 1988; E. Reissner, Freie und erzwungene Torsionsschwingungen des elastischen Halbraumes, Ingenieur-Archiv 8 (1937) 229–245) or by considering couplings with electric and magnetic fields for different types of materials (Maugin, 1988). The theory employed here is the simplest possible version of Mindlin's (R.D. Mindlin, Micro-structure in linear elasticity, Arch. Rat. Mech. Anal. 16 (1964) 51–78) generalized linear elasticity. A simple wave-propagation analysis based on Hankel transforms and complex-variable theory was done in order to determine the conditions for the existence of the torsional surface motions and to derive dispersion curves and cut-off frequencies. Also, we notice that, up to date, no other generalized linear continuum theory (including the integral-type non-local theory) has successfully been proposed to predict torsional surface waves in a homogeneous half-space.

1. Introduction

The classical theory of linear elasticity fails to predict the propagation of free *torsional* surface waves in a *homogeneous* half-space [1–3,4,6]. The criterion for free *surface* waves is that the displacement decays exponentially with distance from the traction-free surface of the body. An analogous situation is also encountered in the case of SH waves (i.e. anti-plane shear wave motions). On the contrary, both plane stress/strain and axisymmetric surface waves of the *Rayleigh* type are predicted by the classical theory of linear elasticity (see e.g. [6–8]). However, surface waves of the *shear* type (i.e. torsional and SH) are known to exist in the nature (see e.g. [3,9,10]). As Maugin [3] points out this paradox within classical linear elasticity is only resolved by properly

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perturbing the boundary conditions in the problem. In addition, Maugin [3] discerns between 'mechanical' perturbations (Love waves, inhomogeneous substrate, zero-thickness interface, and surface curvature and roughness) and 'non-mechanical' perturbations (couplings of the acoustic problem with non-mechanical fields such as electric and magnetic fields).

In view of the works of Agmon et al. [11,12], Vekua [13] and Thompson [14], the situation concerning non-existence of torsional and SH surface waves in a homogeneous (isotropic or anisotropic) half-space, within the context of classical linear elasticity, is translated mathematically to the violation of the pertinent *complementing* (or *consistency*) condition in a semi-infinite domain for the system consisting of a Helmholtz partial differential equation (governing time-harmonic torsional motions in cylindrical-polar coordinates and SH motions in Cartesian coordinates), a zero Neumann boundary condition at the traction-free surface and a finiteness condition at infinity. In general, the complementing or consistency condition on boundary data in a boundary-value problem is a suitability condition of them to the governing differential equation (or to the system of governing differential equations) (see e.g. [11,12,15,16]). This condition may have a rather simple form when bounded domains are considered [16,17] but has not an explicit form for a semi-infinite domain [11,12].

In particular, within the classical linear elasticity theory, Thompson [14] showed that the complementing condition implies that all surface waves propagate with non-zero velocity. Therefore, this condition is obviously satisfied when both dilatational and shear deformations are allowed to take place in the half-space, and thus, Rayleigh surface waves are predicted by the classical theory in the cases of plane stress/strain and general axisymmetric motions. However, the complementing condition is not satisfied in the cases of *torsion* and *anti-plane shear.* In conclusion, the classical linear elasticity theory exhibits a mathematical defect (i.e. ill-posedness) in some cases where a simple zero Neumann condition does not conform to the governing Helmholtz equation in a half-space domain (see [13] for a proof of the latter statement). Here, in the spirit of the analysis by Maugin [3] (i.e. by perturbing the *strong* boundary condition of the classical theory), we consider a generalized linear continuum theory to provide a *regularization* of the aforementioned ill-posed torsional problem of classical elasticity. The corresponding case of SH waves was also recently treated by the present authors [18].

The continuum theory employed in the present study was introduced by Vardoulakis and Sulem [19] as an effective and simple version of Mindlin's [5] general linear elasticity theory with micro-structure. Mindlin begins with the very general concept of an elastic continuum each point (or element) of which is in itself a *deformable* medium (i.e. a micro-medium embedded in the macro-medium). He introduced therefore a continuum with *unit cells* (micro-media) in order to model periodic structures like those of crystal lattices, molecules of a polymer, crystallites of a polycrystal or grains of a granular material. Then, appropriate kinematical quantities are defined to describe geometrical changes in both the macro- and micro-medium. As Toupin [20] notes if each micro-medium is constrained to deform homogeneously, Mindlin's continuum reverts to Ericksen and Truesdell's [21] oriented continuum with deformable directors. Also, Mindlin [5] demonstrated that his theory contains the linear equations of the Cosserat theory [22] and couple stress theory [23,24] as a special case.

Interesting reviews and applications of the above generalized continuum theories and some simplified versions of them are contained, for instance, in the works of Green [25], Muki and Sternberg [26,27], Weitsman [28,29], Jaunzemis [30], Mindlin and Eshel [31], Herrmann and Achenbach [32], Eringen [33,34], Germain [35], Maugin [36], Anderson and Lakes [37,38], and Aifantis [39]. It is evident from these works that Mindlin's [5] theory is one of the most complete *linear* generalized continuum theories. However, in its general form, it involves an enormously large number of material constants (903 constants!). A considerable simplification is gained by taking the so-called relative deformation equal to zero [5]. Further, the work of Vardoulakis and co-workers [19,40,41,42] introduces a simple expression for the strain-energy density (which involves the standard Lame constants and only two additional constants) and provides a physical meaning to the extra terms. The additional constants are 'internal' characteristic lengths and their appearance follows the fact that the theory introduces dependence on *higher-order* deformation gradients. It is noticed that the presence of these characteristic lengths accentuates the importance of size effects, and contrary to the classical theory, affords their incorporation into stress analysis. Finally, by utilizing this theory, Georgiadis and Vardoulakis [43] attacked the anti-plane analogue of Lamb's [6,8] problem (i.e. a half-space under a dynamic concentrated load at the surface) and found that, as opposed to the corresponding result of the classical elasticity theory, the displacement was bounded even at the point of application of the load.

Therefore, in the present study free time-harmonic torsional motions are considered for a homogeneous gradient-elastic half-space. The Hankel transform, elements of complex-variable theory and a parametric analysis of the pertinent dispersion equation are utilized in order to determine the conditions for the existence of torsional surface waves. Our analysis can be useful in wave-propagation studies (e.g. in relation with Non-Destructive Testing and Evaluation) for materials exhibiting *surface effects*. Such effects were discussed by Weitsman [29], Maugin [3,36], Anderson and Lakes [37,38], Vardoulakis and Sulem [19], and Exadaktylos and Vardoulakis [42], among others, and may occur because of damage of surface layers or because of an incomplete structure of edge cells near the surface of solids. Finally, as regards *non-linear* aspects of surface waves, we refer to the recent work by Maugin and co-workers (see e.g. [44,45]).

2. Basic preliminaries

In this section we briefly present the basic ideas and equations which describe the dynamics of *gradient-elastic* (*dipolar* or *grade two* elastic) homogeneous materials *without* couple-stress effects according to Mindlin [5], Green [25], Jaunzemis [30], and Vardoulakis and co-workers [19,40–42]. The point of departure is the following expressions for the strain-energy density W and kinetic-energy density T in a 3D continuum (which is composed wholly of unit cells), with respect to a Cartesian coordinate system $Ox_1x_2x_3$

$$W = \tfrac{1}{2}c_{pqsj}\varepsilon_{pq}\varepsilon_{sj} + \tfrac{1}{2}d_{pqsj\ell m}\kappa_{pqs}\kappa_{j\ell m} + f_{pqsj\ell}\kappa_{pqs}\varepsilon_{j\ell}, \tag{1}$$

$$T = \tfrac{1}{2}\rho\dot{u}_p\dot{u}_p + \tfrac{1}{6}\rho h^2\left(\partial_p\dot{u}_q\right)\left(\partial_p\dot{u}_q\right), \tag{2}$$

where $(c_{pqsj}, d_{pqsjlm}, f_{pqsjl})$ are tensors of material constants, ρ is the mass density, $2h$ is the internal characteristic length of the *structured continuum*, u_p is the displacement vector, $\varepsilon_{pq} = (1/2)(\partial_p u_q + \partial_q u_p)$ is the usual linear strain tensor with $\partial_p \equiv \partial/\partial x_p$, $\kappa_{pqs} = \partial_p\varepsilon_{qs} = \partial_p\varepsilon_{sq}$, $(\dot{\ })$ denotes time differentiation, and the Latin indices span the range (1,2,3).

Then, appropriate definitions for the stresses follow from the variation of W

$$\tau_{pq} = \frac{\partial W}{\partial\varepsilon_{pq}}, \tag{3a}$$

$$m_{pqs} = \frac{\partial W}{\partial\kappa_{pqs}} \equiv \frac{\partial W}{\partial(\partial_p\varepsilon_{qs})}, \tag{3b}$$

where τ_{pq} is the (symmetric) Cauchy stress tensor and $m_{pqs} = m_{psq}$ is the double (or dipolar) stress tensor. The latter tensor follows from the notion of *multipolar forces*, which naturally arise from the following expansion of the mechanical power M [30] $M = F_p\dot{u}_p + F_{pq}(\partial_p\dot{u}_q) + F_{pqs}(\partial_p\partial_q\dot{u}_s) + \cdots$, where F_p are the usual forces (monopolar forces) within the classical continuum mechanics (monopolar continuum mechanics), and $(F_{pq}, F_{pqs}, \ldots)$ are referred to as multipolar forces (double forces, triple forces and so on). Thus, the resultant force on an ensemble of particles (material cells) can be viewed as being decomposed into *external* and *internal* forces with the latter ones being self-equilibrating. However, these self-equilibrating forces (which are multipolar forces) produce *non-vanishing* stresses (double stresses, triple stresses and so on).

Further, from Hamilton's principle and variational considerations on (1) and (2), where the only independent variation is δu_p [5], one obtains the following *equations of motion* (in the absence of body forces) and the *traction* boundary conditions along a smooth boundary

$$\partial_p(\tau_{pq} - \partial_s m_{spq}) = \rho\ddot{u}_q - \frac{\rho h^2}{3}(\partial_{pp}\ddot{u}_q), \tag{4}$$

$$n_r\tau_{rs} - n_q n_r n_l \partial_l m_{qrs} - \left(n_r\left(\delta_{ql} - n_q n_l\right)\partial_l + n_q\left(\delta_{rl} - n_r n_l\right)\partial_l\right) m_{qrs}$$
$$+\left(n_q n_r\left(\delta_{\ell j} - n_\ell n_j\right)\partial_j n_l - \left(\delta_{rl} - n_r n_\ell\right)\partial_\ell n_q\right) m_{qrs} + \frac{\rho h^2}{3} n_r\left(\partial_r \ddot{u}_s\right) = P_s, \tag{5a}$$

$$n_q n_r m_{qrs} = R_s\ , \tag{5b}$$

where n_s is the outward unit normal to the smooth boundary, δ_{pl} is the Kronecker delta, P_s is the surface force per unit area, and R_s is the surface double force (without moment) per unit area. Force systems like R_s have been considered by Love ([46], p. 187) and Jaunzemis ([30], p. 230). In addition to (5), *displacement* boundary conditions were also derived by Vardoulakis and Sulem [19] but are omitted here since these are not relevant to our specific problem.

Alternatively, from the analysis of Georgiadis and Vardoulakis [47] based on the momenta balance, the following equations are obtained:

$$\partial_p\left(\tau_{pq} + \alpha_{pq}\right) = \rho \ddot{u}_q\ , \tag{6a}$$

$$\partial_p m_{pqs} + \alpha_{qs} = \frac{\rho h^2}{3}\left(\partial_q \ddot{u}_s\right)\ , \tag{6b}$$

$$n_q(\tau_{qs} + \alpha_{qs}) = P_s, \tag{7a}$$

$$n_p n_q m_{pqs} = R_s\ , \tag{7b}$$

where $\alpha_{pq} = -\alpha_{qp}$ is the so-called relative stress tensor (which is workless in the present case of double forces without moment). Obviously, Eqs. (6a) and (6b) can be written as the *single* equation (4). Finally, the total stress σ_{pq} (which is asymmetric) is defined through

$$\sigma_{pq} = \tau_{pq} + \alpha_{pq}. \tag{8}$$

Notice that the relative stress α_{pq} can explicitly be obtained only by (6b) and (3b). Then, the total stress may result from (3a) and (8).

Now, the simplest possible form of the strain-energy density function in (1) is considered [19,40–42]

$$W = \tfrac{1}{2}\lambda\varepsilon_{pp}\varepsilon_{qq} + \mu\varepsilon_{pq}\varepsilon_{qp} + \tfrac{1}{2}\lambda c\left(\partial_s\varepsilon_{pp}\right)\left(\partial_s\varepsilon_{qq}\right) + \mu c\left(\partial_s\varepsilon_{pq}\right)\left(\partial_s\varepsilon_{pq}\right) + \tfrac{1}{2}\lambda b_s\partial_s\left(\varepsilon_{pp}\varepsilon_{qq}\right)$$
$$+\mu b_s\partial_s\left(\varepsilon_{pq}\varepsilon_{qp}\right), \tag{9}$$

which, along with the definitions (3), leads to the following *constitutive relations*:

$$\tau_{pq} = \left(\lambda\delta_{pq}\varepsilon_{ss} + 2\mu\varepsilon_{pq}\right) + b_s\partial_s\left(\lambda\delta_{pq}\varepsilon_{ss} + 2\mu\varepsilon_{pq}\right), \tag{10}$$

$$m_{spq} = \left(b_s + c\partial_s\right)\left(\lambda\delta_{pq}\varepsilon_{jj} + 2\mu\varepsilon_{pq}\right), \tag{11}$$

where λ and μ are the standard Lame's constants, c is the gradient coefficient (positive constant with dimensions of square length), $b_s \equiv b\nu_s$ is a material director, with $\nu_s\nu_s = 1$ and b being another material parameter with dimensions of length. Vardoulakis and co-workers [19,40–42] explained the meaning of the last two terms in the r.h.s. of (9) as surface-energy terms by writing them under the form (using the Green–Gauss theorem)

$$\int_{(V)} \partial_s[b_s(\tfrac{1}{2}\lambda\varepsilon_{pp}\varepsilon_{qq} + \mu\varepsilon_{pq}\varepsilon_{qp})]\mathrm{d}V = b\int_{(S)}(\tfrac{1}{2}\lambda\varepsilon_{pp}\varepsilon_{qq} + \mu\varepsilon_{pq}\varepsilon_{qp})(\nu_s n_s)\,\mathrm{d}S, \tag{12}$$

where $\int_{(V)}$ denotes integration over the volume V of the body, $\int_{(S)}$ integration over the surface S enclosing V, and n_s is the outward unit normal to the smooth boundary. In the present study, we consider the particular case $\nu_s \equiv -n_s$ which seems natural for a body in the form of a half-space and which corresponds physically to a weakening or strengthening (this depends upon the choice of a positive or a negative b, respectively) of the body along the direction normal to the surface. In this way, *surface effects* (e.g. material decohesion or surface tension) can *directly* be taken into account in the material constitutive behavior by introducing some *anisotropy* in the material response.

In summary, Eqs. (4), (10) and (11) are the governing equations for the gradient elasticity theory with no couple stresses in a Cartesian $Ox_1x_2x_3$ coordinate system. Combining these equations leads to the field equation of the problem. Pertinent *uniqueness* theorems have been proved for the more general Mindlin's theory [31,48,49] on the basis of the constraint of a positive definite strain-energy density. The latter restriction requires, in turn, the following inequalities for the material constants appearing in the theory employed here [47]

$$(3\lambda + 2\mu) > 0, \tag{13a}$$

$$\mu > 0, \tag{13b}$$

$$c > 0, \tag{13c}$$

$$-1 < (b/c^{1/2}) < 1. \tag{13d}$$

Below, we also provide a *physical* argument for the necessity of (13c). In addition, *stability* for the field equation was proved in [47] and to accomplish this the condition $c > 0$ is a necessary one.

Finally, an estimation for the gradient coefficient c was provided by Altan and Aifantis [50] within a simpler gradient elasticity theory (i.e. omitting the surface-energy term). They give $c \cong (0.25h)^2$, where $2h$ is the characteristic length of the material cell.

3. Governing equations for torsional motions in a half-space

Attention now is directed to the *torsional* dynamic motions in a gradient-elastic half-space with surface energy. With respect to a system of cylindrical coordinates (r, θ, z) having unit base vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$, the half-space occupies the region $(0 \leq r < \infty, z \geq 0)$, see Fig. 1. In this case the material director associated with the surface-energy term takes the form

$$b_r = b_\theta = 0, \quad b_z \equiv b \neq 0. \tag{14}$$

In order to transform the governing equations (4), (10) and (11) into equations in cylindrical coordinates, we first write the former ones in a *tensorial* form and then use *scale factors*, *physical components* and the *invariance* of the form of certain operators (see e.g. [51]). If we omit the terms accounted for dilatational deformation in Eqs.

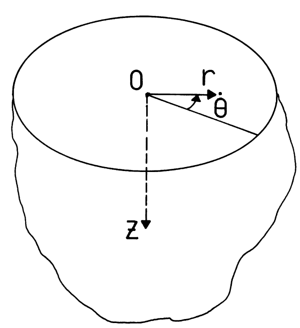

Fig. 1. An elastic half-space in a torsional state.

(4), (10) and (11) (notice that in the torsional case only shear motions exist), these equations have the following tensorial form:

$$\nabla \cdot \boldsymbol{\tau} - \nabla \cdot (\nabla \cdot \mathbf{m}) = \rho \ddot{\boldsymbol{u}} - I (\nabla \cdot (\nabla \ddot{\boldsymbol{u}}), \tag{15}$$

$$\boldsymbol{\tau} = 2\mu \boldsymbol{\varepsilon} + 2\mu b \mathbf{e}_z \cdot (\nabla \boldsymbol{\varepsilon}), \tag{16}$$

$$\mathbf{m} = 2\mu \left[b\mathbf{e}_z \otimes \boldsymbol{\varepsilon} + c\,(\nabla \boldsymbol{\varepsilon}) \right], \tag{17}$$

whereas Eq. (6b) is written as

$$\boldsymbol{\alpha} = -\nabla \cdot \mathbf{m} + I\,(\nabla \ddot{\boldsymbol{u}}), \tag{18}$$

where $I = \rho h^2/3$ is the micro-inertia coefficient.

Further, from the general exposition of Malvern [51] we take the pertinent scale factors as

$$h_r = 1, \quad h_\theta = r, \quad h_z = 1, \tag{19}$$

whereas the components of the displacement vector are

$$u_r = u_z = 0, \tag{20a}$$

$$u_\theta \equiv u\,(r, z, t) \neq 0. \tag{20b}$$

Of notice also are the following results:

$$(\partial \mathbf{e}_r/\partial r) = (\partial \mathbf{e}_r/\partial z) = 0, \quad (\partial \mathbf{e}_r/\partial \theta) = \mathbf{e}_\theta, \quad (\partial \mathbf{e}_\theta/\partial r) = (\partial \mathbf{e}_\theta/\partial z) = 0,$$

$(\partial \mathbf{e}_\theta/\partial \theta) = -\mathbf{e}_r$, $(\partial \mathbf{e}_z/\partial r) = (\partial \mathbf{e}_z/\partial \theta) = (\partial \mathbf{e}_z/\partial z) = 0$. Finally, the strain tensor in the torsional case has the physical components (see e.g. [51])

$$\gamma_{r\theta} \equiv 2\varepsilon_{r\theta} = \frac{\partial u}{\partial r} - \frac{u}{r}, \tag{21}$$

$$\gamma_{z\theta} \equiv 2\varepsilon_{z\theta} = \frac{\partial u}{\partial z}. \tag{22}$$

In view of the aforementioned results, the governing equations (15)–(17) take the following form

$$\begin{aligned}\left(\frac{\partial \tau_{r\theta}}{\partial r} + \frac{2}{r}\tau_{r\theta} + \frac{\partial \tau_{z\theta}}{\partial z}\right) - \left(\frac{\partial^2 m_{rr\theta}}{\partial r^2} + \frac{\partial^2 m_{zr\theta}}{\partial r \partial z} + \frac{\partial^2 m_{rz\theta}}{\partial z \partial r} + \frac{\partial^2 m_{zz\theta}}{\partial z^2} + \frac{3}{r}\frac{\partial m_{rr\theta}}{\partial r} + \frac{1}{r}\frac{\partial m_{rz\theta}}{\partial z} + \frac{2}{r}\frac{\partial m_{zr\theta}}{\partial z}\right.\\ \left. + \frac{1}{r}\frac{\partial m_{\theta rr}}{\partial r} - \frac{1}{r}\frac{\partial m_{\theta\theta\theta}}{\partial r} + \frac{1}{r}\frac{\partial m_{\theta zr}}{\partial z} + \frac{1}{r^2}m_{\theta rr} - \frac{1}{r^2}m_{\theta\theta\theta} + \frac{1}{r^2}m_{r\theta r}\right)\\ = \rho \ddot{u} - I\left(\frac{\partial^2 \ddot{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \ddot{u}}{\partial r} - \frac{1}{r^2}\ddot{u} + \frac{\partial^2 \ddot{u}}{\partial z^2}\right),\end{aligned} \tag{23}$$

$$\tau_{r\theta} \equiv \tau_{\theta r} = \mu \gamma_{r\theta} + \mu b \frac{\partial \gamma_{r\theta}}{\partial z}, \tag{24a}$$

$$\tau_{z\theta} \equiv \tau_{\theta z} = \mu \gamma_{z\theta} + \mu b \frac{\partial \gamma_{z\theta}}{\partial z}, \tag{24b}$$

$$m_{rr\theta} \equiv m_{r\theta r} = \mu c \frac{\partial \gamma_{r\theta}}{\partial r}, \tag{25a}$$

$$m_{rz\theta} \equiv m_{r\theta z} = \mu c \frac{\partial \gamma_{z\theta}}{\partial r}, \tag{25b}$$

$$m_{zr\theta} \equiv m_{z\theta r} = \mu b\gamma_{r\theta} + \mu c\frac{\partial \gamma_{r\theta}}{\partial z}, \tag{25c}$$

$$m_{zz\theta} \equiv m_{z\theta z} = \mu b\gamma_{z\theta} + \mu c\frac{\partial \gamma_{z\theta}}{\partial z}, \tag{25d}$$

$$m_{\theta\theta\theta} = \frac{2\mu c}{r}\gamma_{r\theta}, \tag{25e}$$

$$m_{\theta rr} = -\frac{2\mu c}{r}\gamma_{r\theta}, \tag{25f}$$

$$m_{\theta zr} \equiv m_{\theta rz} = -\frac{\mu c}{r}\gamma_{z\theta}. \tag{25g}$$

Inserting now (24a), (24b) and (25a)–(25g) in (23) and using (21) and (22), we obtain the *field equation* for torsional wave motions in a gradient-elastic half-space

$$c\nabla^2\left(\nabla^2 u - \frac{u}{r^2}\right) - \left(1 + \frac{c}{r^2}\right)\left(\nabla^2 u - \frac{u}{r^2}\right) = \frac{I}{\mu}\left(\nabla^2 \ddot{u} - \frac{\ddot{u}}{r^2}\right) - \frac{1}{V^2}\ddot{u}, \tag{26}$$

where $\nabla^2 \equiv \left[\left(\partial^2/\partial r^2\right) + (1/r)\,(\partial/\partial r) + \left(\partial^2/\partial z\right)\right]$ is the Laplace operator in cylindrical polar coordinates, and $V \equiv (\mu/\rho)^{1/2}$ is the shear-wave velocity in the *absence* of gradient effects (i.e. in classical elasticity).In the absence of gradient effects, i.e. when $c=0$, Eq. (26) degenerates into the standard wave equation of the second order governing torsional motions (see e.g. [6]). The *dispersive* character of (26) can immediately be seen by writing it as a wave equation with a non-constant coefficient (or better, with a coefficient in the form of an operator): $\left(c\nabla^2 - \left[1 + (c/r^2)\right]\right)\left(\nabla^2 u - (u/r^2)\right) = (I/\mu)(\nabla^2\ddot{u} - \ddot{u}/r^2) - (1/V^2)\ddot{u}$. Finally, (26) implies that the propagation of a wave of the form $u(r, z=0, t) = u_0 \cdot \xi \cdot J_1(\xi r) \cdot \exp(-\mathrm{i}\omega t)$ satisfies $C_{\mathrm{ph}}^2 = V^2(g + c\xi^2)$ and $\omega^2 = V^2\xi^2(g + c\xi^2)$, where $C_{ph} = \omega/\xi$ is the phase velocity, ξ is the propagation wave number (taken to be a real quantity), ω is the frequency of the cylindrical wave (also taken to be a real quantity), and u_0 is a constant amplitude. Then, one may observe that the condition $c>0$ (i.e. relation (13c)) *always* secures, viz. independently of ξ, a *real* propagation velocity, a fact which is most relevant to wave propagation problems.

Writing the governing equations for the present problem is completed by obtaining the total stresses $\sigma_{r\theta} \equiv \tau_{r\theta} + \alpha_{r\theta}$ and $\sigma_{z\theta} \equiv \tau_{z\theta} + \alpha_{z\theta}$ (see Eq. (8)). Indeed, the relative stresses result from (18) as

$$\alpha_{r\theta} = I\left(\frac{\partial \ddot{u}}{\partial r}\right) - \mu c\left(\frac{\partial^3 u}{\partial r^3} - \frac{3}{r^2}\frac{\partial u}{\partial r} + \frac{3}{r^3}u + \frac{\partial^3 u}{\partial z^2 \partial r} - \frac{1}{r}\frac{\partial^2 u}{\partial z^2}\right) - \mu b\left(\frac{\partial^2 u}{\partial z \partial r} - \frac{1}{r}\frac{\partial u}{\partial z}\right), \tag{27a}$$

$$\alpha_{z\theta} = I\left(\frac{\partial \ddot{u}}{\partial z}\right) - \mu c\left(\frac{\partial^3 u}{\partial r^2 \partial z} - \frac{1}{r}\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r^2}\frac{\partial u}{\partial z} - \frac{\partial^3 u}{\partial z^3}\right) - \mu b\frac{\partial^2 u}{\partial z^2}, \tag{27b}$$

and in view also of (24a) and (24b), we finally obtain

$$\sigma_{r\theta} = I\left(\frac{\partial \ddot{u}}{\partial r}\right) + \mu\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) - \mu c\left(\nabla^2 - \frac{4}{r^2}\right)\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right), \tag{28a}$$

$$\sigma_{z\theta} = I\left(\frac{\partial \ddot{u}}{\partial z}\right) + \mu\left(\frac{\partial u}{\partial z}\right) - \mu c\left[\left(\nabla^2 - \frac{1}{r^2}\right)\left(\frac{\partial u}{\partial z}\right)\right]. \tag{28b}$$

Eqs. (25a)-(25g), (28a), and (28b) are the constitutive equations for the gradient-elastic response of a half-space under torsional loading and Eq. (26) is the corresponding field equation in the general *transient* case. It is worthwhile noticing that the surface-energy material constant b does not appear in Eqs. (28a) and (28b) relating total stresses and displacement gradients and also in Eq. (26); it only enters the problem through boundary conditions involving double stresses. In the sequel, a *steady state* is considered where, as is well-known (see e.g. [6]), the displacement varies in the following time-harmonic manner:

$$u(r, z, t) = u(r, z) \cdot \exp(-\mathrm{i}\omega t), \tag{29}$$

where $\mathrm{i} \equiv (-1)^{1/2}$ and ω is the frequency. The above ‘decomposition’ reduces (26) to the form

$$c\left(\nabla^2 - \frac{1}{r^2}\right)\left(\nabla^2 u - \frac{u}{r^2}\right) - g\left(\nabla^2 u - \frac{u}{r^2}\right) - k^2 u = 0, \tag{30}$$

and the stress–strain relation (28a) and (28b) to the form

$$\sigma_{r\theta} = \mu g\left(\frac{\partial u}{\partial r}\right) - \mu\frac{u}{r} - \mu c\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right), \tag{31a}$$

$$\sigma_{z\theta} = \mu g\left(\frac{\partial u}{\partial z}\right) - \mu c\left[\left(\nabla^2 - \frac{1}{r^2}\right)\left(\frac{\partial u}{\partial z}\right)\right], \tag{31b}$$

where $k = \omega/V$ and $g = 1-(\omega^2 I/\mu)$. In what follows, as is standard in this type of problems, it is implied that all field quantities are to be multiplied by the time-harmonic factor $\exp(-\mathrm{i}\omega t)$ and that the real part of the resulting expression is to be taken. The function $u(r,z)$ obeying the fourth-order PDE (30) is called *metaharmonic* function and can be written as the sum of two arbitrary solutions of two pertinent Helmholtz equations ([13], p. 206).

4. Integral-transform analysis

In order to suppress the r-dependence in the governing equations and the boundary conditions, the *Hankel transform* of order one defined as follows is employed (see e.g. [52])

$$f^*(\xi, z) = \int_0^\infty f(r, z) \cdot J_1(\xi r) \cdot r\,\mathrm{d}r, \tag{32a}$$

$$f(r, z) = \int_0^\infty f^*(\xi, z) \cdot J_1(\xi r) \cdot \xi\,\mathrm{d}\xi, \tag{32b}$$

where $J_1(\ )$ is the Bessel function of the first kind of order one. Under the operation (32a) and assuming those conditions for $u(r,z)$ stated, e.g., in Section 154 of [52], Eq. (30) is transformed into the *ordinary* differential equation

$$c\frac{\mathrm{d}^4 u^*}{\mathrm{d}z^4} - \left(2c\xi^2 + g\right)\frac{\mathrm{d}^2 u^*}{\mathrm{d}z^2} + \left(c\xi^4 + g\xi^2 - k^2\right)u^* = 0, \tag{33}$$

and we may extend, by analytic continuation, the range of the transform variable ξ into the whole complex plane cut with pertinent branch lines.

Now, (33) has the following general solution:

$$u^*(\xi, z) = C_1(\xi) \cdot \mathrm{e}^{X_1 z} + C_2(\xi) \cdot \mathrm{e}^{X_2 z} + C_3(\xi) \cdot \mathrm{e}^{X_3 z} + C_4(\xi) \cdot \mathrm{e}^{X_4 z}, \tag{34a}$$

where $C_1(\xi), \ldots, C_4(\xi)$ are unknown functions, and $(X_1, \ldots X_4)$ are the roots of the corresponding *characteristic* quartic algebraic equation. However, in view of the fact that these roots occur in pairs as $X_1 = -X_2$ and $X_3 = -X_4$, a *bounded* solution as $z \to \infty$ (i.e. one that satisfies the *finiteness* condition at infinity, see e.g. [13]) has the form

$$u^*(\xi, z) = B(\xi) \cdot \mathrm{e}^{-\beta z} + C(\xi) \cdot \mathrm{e}^{-\gamma z} \quad \text{for} \quad z \geq 0, \tag{34b}$$

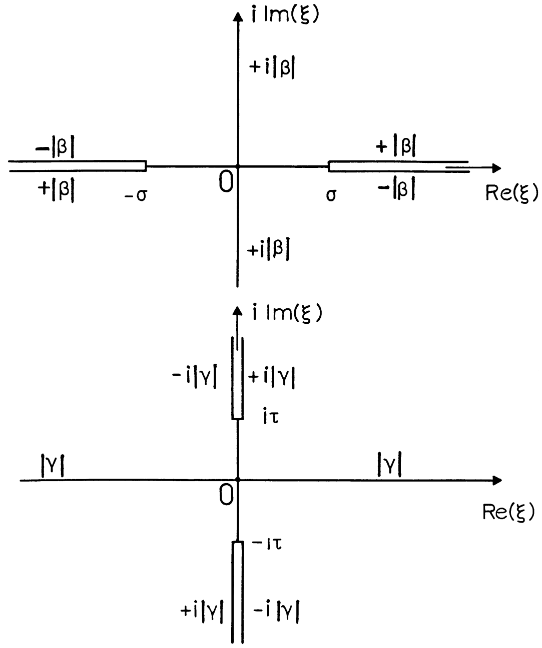

Fig. 2. The cut complex ξ-plane for the functions $\beta(\xi)$ and $\gamma(\xi)$.

provided that the ξ-plane has been cut appropriately. In this case, $B(\xi)$ and $C(\xi)$ play the role of the unknown functions and (β,γ) are the relevant roots given by

$$\beta \equiv \beta(\xi) = (\xi^2 - \sigma^2)^{1/2}, \tag{35a}$$

with

$$\sigma = \frac{\left[(g^2 + 4ck^2)^{1/2} - g\right]^{1/2}}{(2c)^{1/2}} > 0, \tag{35b}$$

$$\gamma \equiv \gamma(\xi) = (\xi^2 + \tau^2)^{1/2}, \tag{36a}$$

with

$$\tau = \frac{\left[(g^2 + 4ck^2)^{1/2} + g\right]^{1/2}}{(2c)^{1/2}} > 0. \tag{36b}$$

Compatible with the above analysis is taking the *branch cuts* for (β,γ) shown in Fig. 2. Therefore, any inversion of the type (32b) should be performed considering this restriction.

Finally, the transformed components of the total-stress tensor and double-stress tensor entering the boundary conditions are obtained by operating with (32a) on (31b) and (25d) and by taking into account the general solution in (34b)

$$\sigma^*_{z\theta}(\xi, z) = -\mu c\tau^2 \beta B\, \mathrm{e}^{-\beta z} + \mu c\sigma^2 \gamma C\, \mathrm{e}^{-\gamma z} \quad \text{for } z \geq 0, \tag{37}$$

$$m^*_{zz\theta}(\xi, z) = \mu\left[(c\beta - b)\beta B\, \mathrm{e}^{-\beta z} + (c\gamma - b)\gamma C\, \mathrm{e}^{-\gamma z}\right] \quad \text{for } z \geq 0. \tag{38}$$

5. Free torsional surface waves

The criterion for surface waves in this case is that the displacement u decays exponentially with the distance z from the free surface. Such a case for a *homogeneous* half-space is precluded according to the classical elasticity theory [3] but can arise, as it is shown below, within the present gradient elasticity theory (Eqs. (15)–(17)). Indeed, in view of the analysis leading to (34b) we now explore the possibility of *progressive-wave* solutions to (26) having the following form of a distinct harmonic component

$$\bar{u}_s(r,z,t) = \left[B(\xi)\cdot \mathrm{e}^{-|\beta|\cdot z} + C(\xi)\cdot \mathrm{e}^{-|\gamma|\cdot z}\right]\cdot J_1(\xi r)\cdot \xi \cdot \mathrm{e}^{-\mathrm{i}\omega(\xi)\cdot t} \equiv u_s^*(\xi,z)\cdot J_1(\xi r)\cdot \xi\cdot \mathrm{e}^{-\mathrm{i}\omega(\xi)\cdot t}, \qquad (39)$$

where (B,C) represent arbitrary amplitude functions denoting the relative dominance of a particular harmonic component, the propagation wave number ξ is taken to be a real quantity, and (β,γ) defined in (35a) and (36a) are taken to be *real* and *positive* functions. The latter restriction is satisfied if and only if $\sigma < |\xi|$. Taking a *real* wave number excludes the possibility of localized *standing* waves (i.e. leaky or evanescent motions), whereas the frequency at which the wave number (for a particular mode) changes from real to imaginary (or complex) values is the cut-off frequency. Below, we will see what are the conditions for the existence of torsional surface waves, within gradient elasticity, and how cut-off frequencies for these waves can be determined. Finally, we notice that a general surface-wave solution (synthesis) can be derived by (39) as a Hankel inversion integral

$$u_s(r,z,t) = \int_0^\infty u_s^*(\xi,z)\cdot J_1(\xi r)\cdot\xi\cdot \mathrm{e}^{-\mathrm{i}\omega(\xi)\cdot t}\,\mathrm{d}\xi. \qquad (40)$$

Then, the appropriate *dispersion* or *frequency equation* is obtained by enforcing the traction-free boundary conditions along the half-space surface $z=0$. From Eqs. (5a) and (5b) or Eqs. (7a) and (7b), one has

$$\sigma_{z\theta}(r,z=0)=0 \quad \text{for} \quad 0\le r<\infty, \qquad (41a)$$

$$m_{zz\theta}(r,z=0)=0 \quad \text{for} \quad 0\le r<\infty. \qquad (41b)$$

The above conditions are transformed next according to (32a) and the general forms in (37) and (38) are considered but now with $|\beta|=(\xi^2-\sigma^2)^{1/2}$ and $|\gamma|=(\xi^2+\tau^2)^{1/2}$ (with ξ being real and such that $\sigma<|\xi|$). In this way, the following linear homogeneous system results for the unknown functions B and C

$$-\mu c\tau^2\,|\beta|\cdot B + \mu c\sigma^2\,|\gamma|\cdot C = 0, \qquad (42a)$$

$$(c|\beta|-b)|\beta|\cdot B + (c|\gamma|-b)|\gamma|\cdot C = 0, \qquad (42b)$$

which has a non-trivial solution if and only if

$$-c\left[\sigma^2(\xi^2-\sigma^2)^{1/2} + \tau^2(\xi^2+\tau^2)^{1/2}\right] + b\alpha^2 = 0, \qquad (43)$$

with

$$\alpha^2 \equiv \sigma^2+\tau^2 = \frac{(g^2+4ck^2)^{1/2}}{c} > 0. \qquad (44)$$

Eq. (43) is the dispersion equation for the motion of progressive torsional surface waves in a gradient-elastic homogeneous half-space. It is noteworthy that the same equation also prevails in the propagation of SH surface waves [18]. From this equation, dispersion curves were obtained and these will be presented below. Before this, however, the following observations on (43) are in order: (i) Torsional surface waves exist if and only if $c\neq 0$ and $b>0$; the cases ($c=0$) or ($c\neq 0$ and $b=0$) or ($c\neq 0$ and $b<0$) lead to non-existence of such waves. (ii) Eq. (43) being an irrational algebraic equation is a *monomode* dispersion equation and this is in some contrast with the infinity of modes resulting from transcendental equations, which correspond to non-homogeneous models of a half-space supporting torsional surface waves [1,2,4].

In order now to present numerical results in an effective way, the following normalizations are introduced:

$$\xi_d = c^{1/2}\xi, \tag{45a}$$

$$\omega_d = \omega h/3^{1/2}V, \tag{45b}$$

and one may find that $g \equiv 1-\omega_d^2$ by virtue of the latter definition. Also, the phase velocity $C_{\rm ph}$ and wavelength λ are introduced through the standard relations $C_{\rm ph}=\omega/\xi$ and $\lambda=2\pi/\xi$, respectively. Three different relations between h and $c^{1/2}$ are then taken to obtain numerical results, viz. $h=2(3c)^{1/2}$, $(3c)^{1/2}$ and $(1/2)(3c)^{1/2}$, whereas two different values of $b' \equiv b/c^{1/2}$, viz. $b'=0.5$ and 0.9, are considered in each of the previous cases. All results have been obtained with the help of MATHEMATICA.

When $h=2(3c)^{1/2}$, the solution of (43) reads as

$$\xi_d = \left[4\omega_d^6 - 11\omega_d^4 + 11\omega_d^2 - 4 + b'^2(4\omega_d^4 - 6\omega_d^2 + 4) - 2b'\omega_d^2(\omega_d^2 + b'^2 - 1)^{1/2}\right]^{1/2} \times \left(2\left|\omega_d^2 - 1\right|\right)^{-1} \quad \text{for } \omega_d \neq 1, \tag{46a}$$

$$\xi_d = (1+16b'^4)^{1/2}(4b')^{-1} \quad \text{for } \omega_d = 1. \tag{46b}$$

In this case, there is a cut-off frequency at $\omega_d=1.000197$ for $b'=0.5$ and at $\omega_d=0.439796$ for $b'=0.9$. Figs. 3 and 4 depict the variation of the normalized phase velocity C_{ph}/V with the normalized wave number ξh and the normalized wavelength λ/h, respectively.

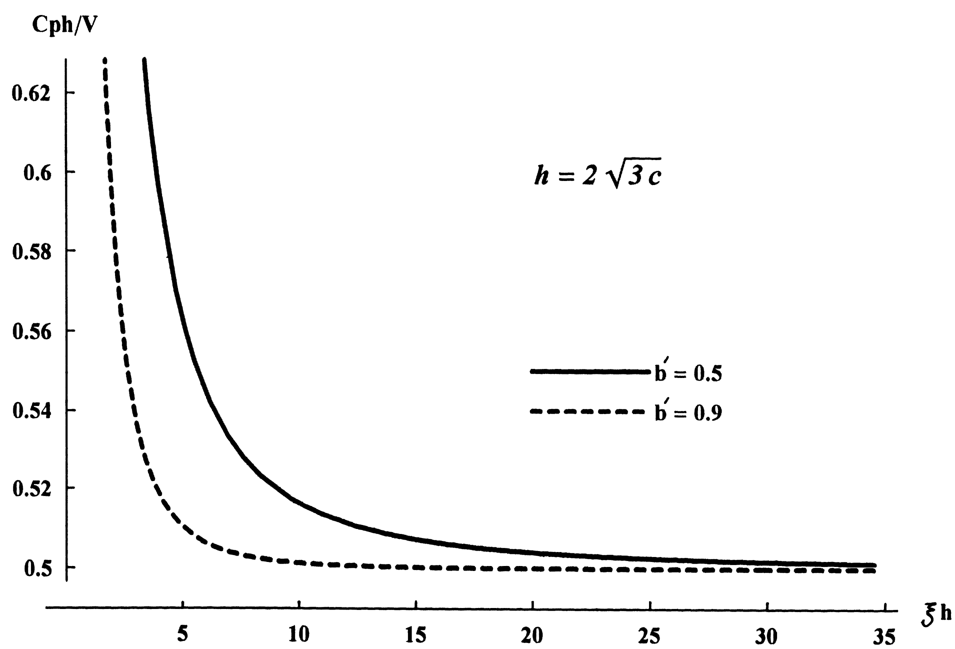

Fig. 3. Dispersion curves for the propagation of torsional surface waves showing the variation of the normalized phase velocity (C_{ph}/V) with the normalized wave number ξh, when $h=2(3c)^{1/2}$.

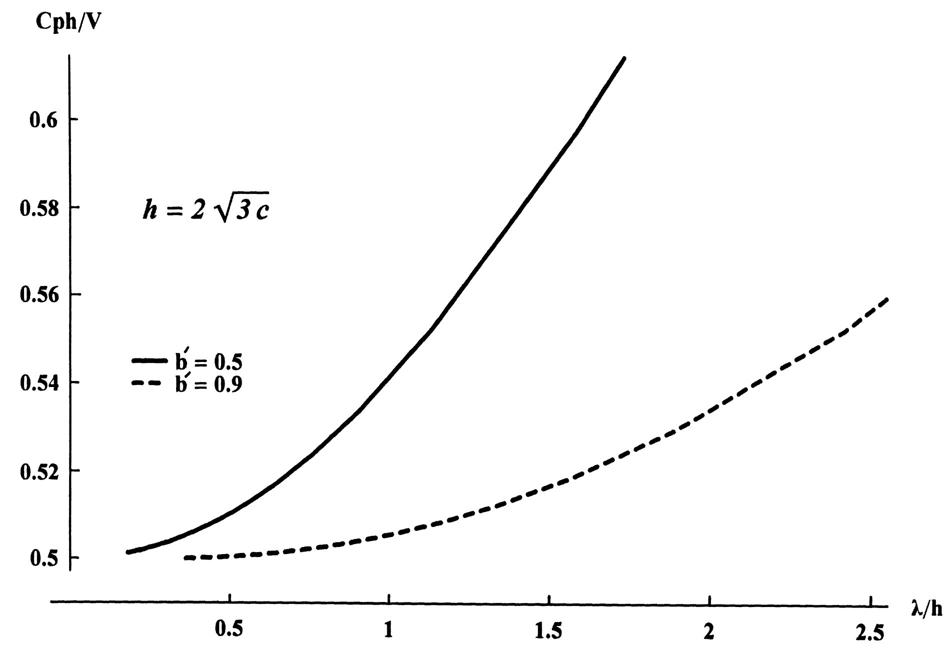

Fig. 4. Dispersion curves for the propagation of torsional surface waves showing the variation of the normalized phase velocity (C_{ph}/V) with the normalized wavelength λ/h, when $h=2(3c)^{1/2}$.

When $h=(3c)^{1/2}$, the solution of (43) is given as

$$\xi_d = \left[\omega_d^6 - 2\omega_d^4 + 2\omega_d^2 - 1 + b'^2(\omega_d^4+1) - 2b'\omega_d^2\,(\omega_d^2+b'^2-1)^{1/2}\right]^{1/2}(|\omega_d^4-1|)^{-1} \quad \text{for} \quad \omega_d \neq 1, \tag{47a}$$

$$\xi_d = (1+4b'^4)^{1/2}(2b')^{-1} \quad \text{for} \quad \omega_d = 1. \tag{47b}$$

Now, there is a cut-off frequency at $\omega_d = 1.732048$ for $b'=0.5$ and at $\omega_d = 0.484316$ for $b'=0.9$. Figs. 5 and 6 depict the variation of C_{ph}/V with ξh and λ/h, respectively.

Finally, when $h=(1/2)(3c)^{1/2}$, the solution of (43) reads as

$$\xi_d = \left[(\omega_d^2-1)(\omega_d^2+1)^2 + b'^2(\omega_d^4+6\omega_d^2+1) - 8b'\omega_d^2\cdot(\omega_d^2+b'^2-1)^{1/2}\right]^{1/2}\left(\left|\omega_d^2-1\right|\right)^{-1}. \tag{48}$$

In this case, a cut-off frequency occurs at $\omega_d = 7.136990$ for $b'=0.5$ and at $\omega_d = 2.492990$ for $b'=0.9$. Figs. 7 and 8 show the variation of C_{ph}/V with ξh and λ/h, respectively.

In all three cases, in the C_{ph}/V vs. ξh graph the cut-off frequencies correspond to the beginning of the graph in the left-hand side of the figure, whereas in the C_{ph}/V vs. λ/h graph the cut-off wave numbers are the ones at which each graph stops in the right-hand side of the figure. Generally, one may observe from the numerical results that the values of the internal length h and the ratio of lengths $b' \equiv b/c^{1/2}$ play a significant role in the form of the dispersion curves and in the occurrence of a cut-off frequency. In particular, we may conclude that the greater the internal length is, the cut-off frequency occurs at a lower value.

Finally, another issue which merits discussion pertains to the form of the dispersion curves. In the case $h=2(3c)^{1/2}$, both curves in Fig. 3 exhibit normal dispersion (i.e. $dC_{ph}/d\xi < 0$), but in the case $h=(3c)^{1/2}$ and $b'=0.9$ (Fig. 5) as the frequency increases anomalous dispersion (i.e. $dC_{ph}/d\xi > 0$) is observed. Also, for even smaller internal lengths h, i.e. in the case $h=(1/2)(3c)^{1/2}$, both curves in Fig. 7 exhibit anomalous dispersion. This finding reminds

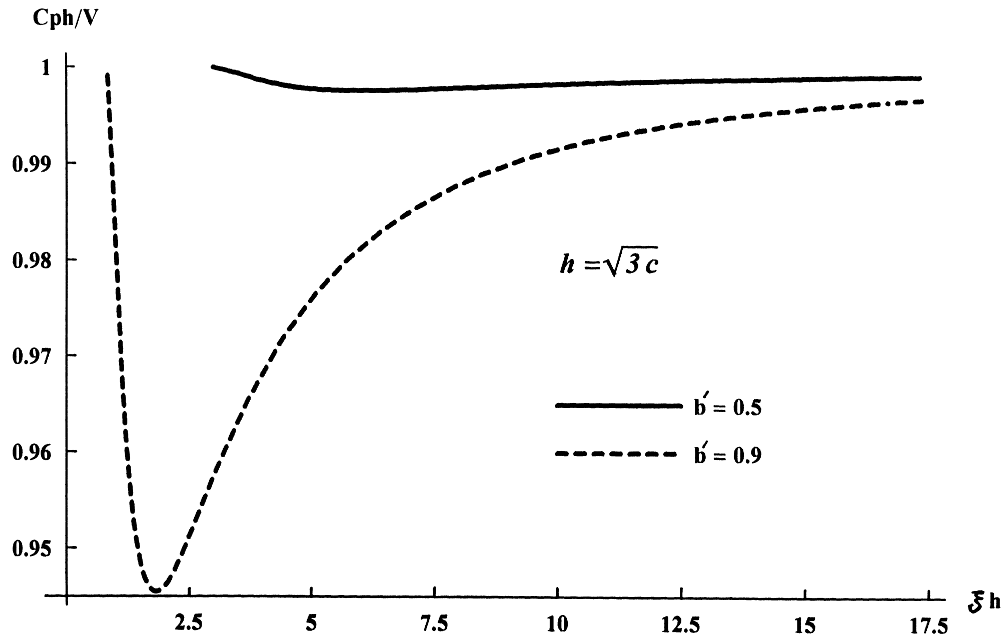

Fig. 5. Dispersion curves for the propagation of torsional surface waves showing the variation of the normalized phase velocity (C_{ph}/V) with the normalized wave number ξh, when $h=(3c)^{1/2}$.

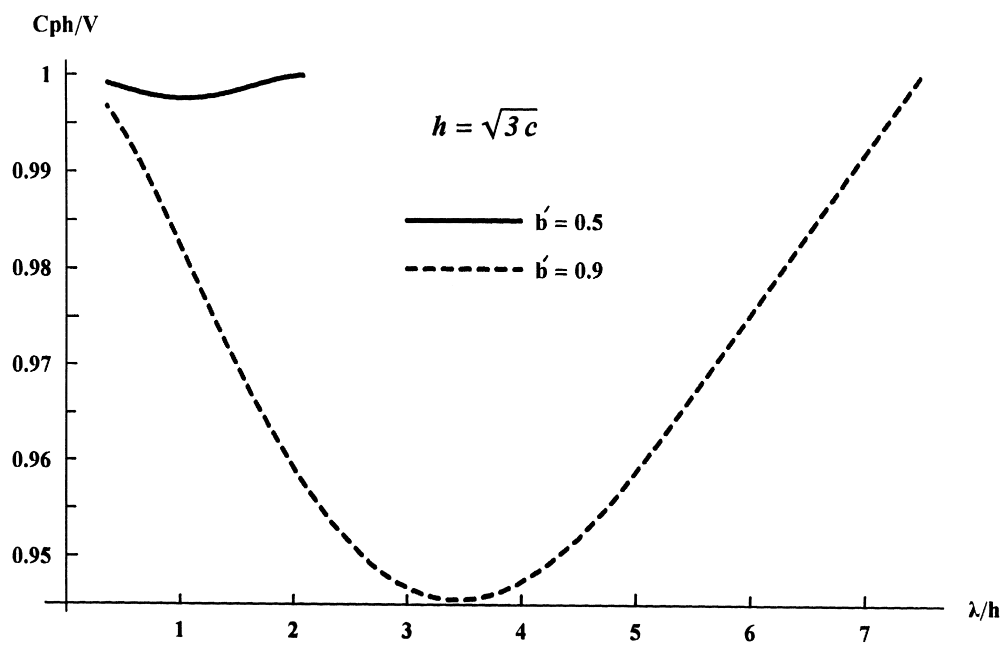

Fig. 6. Dispersion curves for the propagation of torsional surface waves showing the variation of the normalized phase velocity (C_{ph}/V) with the normalized wavelength λ/h, when $h=(3c)^{1/2}$.

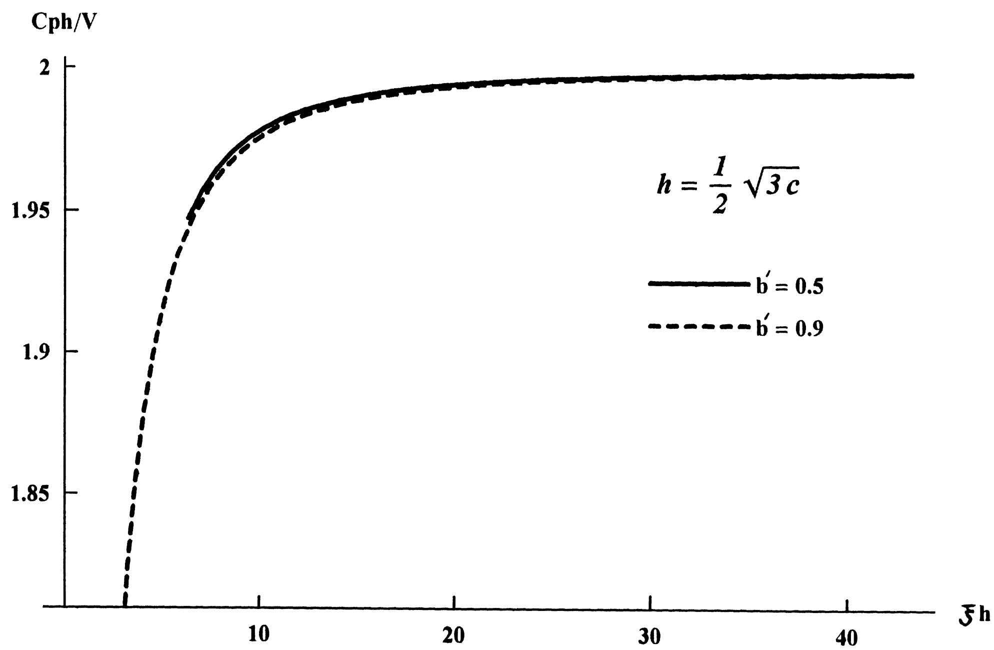

Fig. 7. Dispersion curves for the propagation of torsional surface waves showing the variation of the normalized phase velocity (C_{ph}/V) with the normalized wave number ξh, when $h=(1/2)(3c)^{1/2}$.

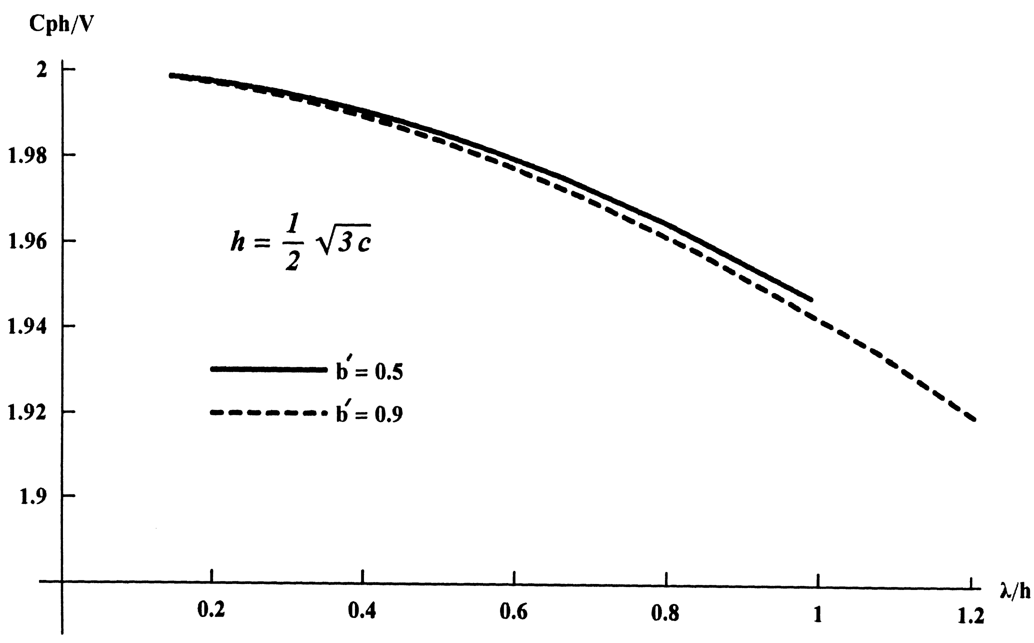

Fig. 8. Dispersion curves for the propagation of torsional surface waves showing the variation of the normalized phase velocity (C_{ph}/V) with the normalized wavelength λ/h, when $h=(1/2)(3c)^{1/2}$.

analogous results for: (i) generalized Rayleigh surface waves in Cu cubic crystals [53], (ii) Stoneley interface waves in a half-space with a superficial layer [54], and (iii) surface waves in liquids that posses surface tension [55].

6. Concluding remarks

The present analysis shows that the mere existence of torsional surface waves gives rise to justifying an extension of classical linear elasticity so as to incorporate: (i) surface-energy terms, which are governed by the material parameter b, (ii) volume strain-gradient terms, which are governed by the material parameter c, and (iii) micro-inertia terms governed by the material parameter h. Furthermore, appropriate measurements revealing the true dispersion relationship could be conducted on the basis of the present model. Our numerical results generally show the dependence of cut-off frequencies and of the character of dispersion (*normal–anomalous*) upon the size of the material unit cell.

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